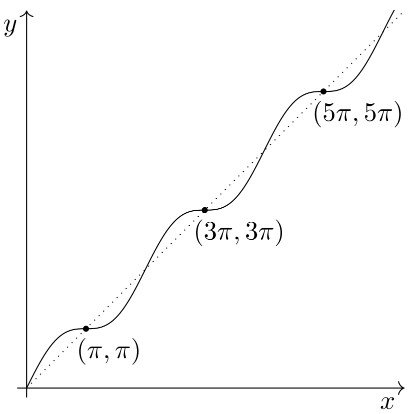
4101. (a) Differentiating,

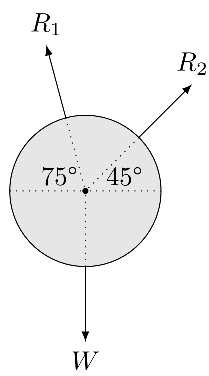
$$\begin{aligned} & \mathrm{f}(x) = \sin x + x \\ \Longrightarrow\; & \mathrm{f}'(x) = \cos x + 1 \\ \Longrightarrow\; & \mathrm{f}''(x) = -\sin x. \end{aligned}$$

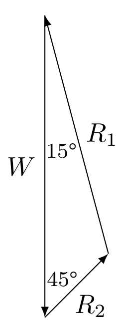
The second derivative is zero and changes sign at the roots of $\sin x$, which are $x = n\pi$ for $n \in \mathbb{Z}$. At these points, the first derivative is alternately 0 and 2. So, half of the points of inflection have gradient 2, and the other half are stationary. There are infinitely many of each.

(b) The curve is the superposition of the linear $y = x$ and the sinusoidal $y = \sin x$. Part (a) tells us that, at $x = n\pi$, the gradient is 0. And, since the range of $\cos x$ is $[-1, 1]$, the range of f' is $[0, 2]$. Putting these facts together, the curve is



4102. The planes are inclined at $\arctan 1 = 45°$ and $\arctan\left(2 - \sqrt{3}\right) = 15°$. The cylinder is smooth, so the contact forces are perpendicular to the planes. The force diagram and triangle of forces are





Using the sine rule,

$$\begin{aligned} R_1 &= \frac{\sin 45°}{\sin 120°} W \equiv \frac{\sqrt{6}}{3} W, \\ R_2 &= \frac{\sin 15°}{\sin 120°} W \equiv \frac{3\sqrt{2} - \sqrt{6}}{6} W. \end{aligned}$$

4103. We integrate by parts. Let $u = x$ and $\frac{dv}{dx} = \mathrm{f}_2(x)$, so that $\frac{du}{dx} = 1$ and $v = \mathrm{f}_1(x)$. The integration by parts formula gives

$$\begin{aligned} & \int x\, \mathrm{f}_2(x)\, dx \\ &= x\, \mathrm{f}_1(x) - \int \mathrm{f}_1(x)\, dx \\ &= x\, \mathrm{f}_1(x) - \mathrm{f}_0(x) + c. \end{aligned}$$

4104. Differentiating,

$$\begin{aligned} & \mathrm{f}(x) = a_1 x + a_2 x^3 + ... + a_k x^{2k+1} \\ \Longrightarrow\; & \mathrm{f}'(x) = a_1 + 3a_2 x^2 + ... + (2k+1) a_k x^{2k} \\ \Longrightarrow\; & \mathrm{f}''(x) = 6a_2 x + ... + (2k+1)(2k) a_k x^{2k-1} \\ & \qquad = x\left(6a_2 + ... + (2k+1)(2k) a_k x^{2k-2}\right). \end{aligned}$$

There is a factor of x, so $\mathrm{f}''(0) = 0$. Furthermore, since $a_2 \neq 0$, this factor is not repeated. So, $\mathrm{f}''(x)$ is zero and changes sign at $x = 0$. And $\mathrm{f}(0) = 0$, so $y = \mathrm{f}(x)$ has a point of inflection at the origin. □

4105. For a triangle to exist, each side must be shorter than the sum of the other two. Assume, without loss of generality, that the common ratio $r \geq 1$. We need $r^2 < 1 + r$. The boundary equation is $r^2 - r - 1 = 0$, which has solution $r = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$. Only the greater of these satisfies $r \geq 1$. So, there are such triangles if

$$1 \leq r < \tfrac{1}{2}\left(1 + \sqrt{5}\right).$$

——— Alternative Method ———

For a triangle to exist, each side must be shorter than the sum of the other two.

- Firstly, assume that $r \geq 1$. We need $r^2 < 1 + r$. The boundary equation is $r^2 - r - 1 = 0$, which has solution $r = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$. The greater of these satisfies $r \geq 1$. So, there are such triangles if

$$1 \leq r < \tfrac{1}{2}\left(1 + \sqrt{5}\right).$$

- Secondly, $0 < r < 1$. We require $1 < r + r^2$. This has boundary equation $r^2 + r - 1 = 0$, which has solution $r = \frac{1}{2}\left(-1 \pm \sqrt{5}\right)$. Only the greater of these is positive. So, there are such triangles if

$$\tfrac{1}{2}\left(1 + \sqrt{5}\right) < r < 1.$$

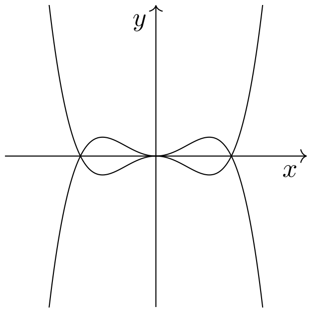
The common ratio cannot be zero/negative, as this would produce zero/negative side lengths. Putting the two options above together:

$$r \in \left(\tfrac{1}{2}\left(-1 + \sqrt{5}\right), \tfrac{1}{2}\left(1 + \sqrt{5}\right)\right).$$

——— Nota Bene ———

Any triangle with lengths in GP can be described with a common ratio $r \geq 1$, as in the first solution above. The lower bound in the second solution, which doesn't seem to appear in the first, in fact does. The two bounds in the second solution are reciprocals of each other, representing the fact that a decreasing GP of side lengths $\{a, ar, ar^2\}$ with $r < 1$ can be reinterpreted, in reverse order, as an increasing GP of side lengths $\{ar^2, ar, a\}$. The common ratio of this sequence is r^{-1}.

4106. All of the points of the given graph have $y \geq 0$. So, they all feature in the new graph. Replacing y with $|y|$ also allows symmetrical negative values of y. So, the graph of $|y| = \left|x^4 - x^2\right|$ is



4107. Let $u = e^x + 3$. Then $du = e^x\,dx$. So,

$$dx = \frac{1}{e^x}\,du = \frac{1}{u-3}\,du.$$

Also, $e^x + 2 = u - 1$. Enacting the substitution,

$$\begin{aligned}&\int \frac{e^x+2}{e^x+3}\,dx \\ =&\int \frac{u-1}{u(u-3)}\,du.\end{aligned}$$

Writing the integrand in partial fractions, this is

$$\begin{aligned}&\tfrac{1}{3}\int \frac{1}{u} + \frac{2}{u-3}\,du \\ =&\ \tfrac{1}{3}\left(\ln|u| + 2\ln|u-3|\right) + c.\end{aligned}$$

Since $u = e^x + 3$ and $u - 3 = e^x$ are both positive, we can get rid of the mod signs, leaving

$$\begin{aligned}&\tfrac{1}{3}\left(\ln(e^x+3) + 2\ln(e^x)\right) + c \\ \equiv&\ \tfrac{1}{3}\ln(e^x+3) + \tfrac{2}{3}x + c.\end{aligned}$$

4108. (a) By the chain rule, the derivative is

$$\frac{dy}{dx} = e^x - e^{-x}.$$

So, at a generic point $(p, e^p + e^{-p})$, the equation of the tangent is

$$y - (e^p + e^{-p}) = (e^p - e^{-p})(x - p).$$

If this passes through the origin, then

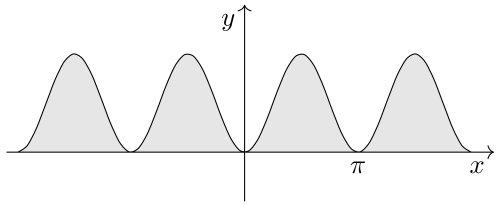
$$\begin{aligned}&-e^p - e^{-p} = (e^p - e^{-p})(-p) \\ \Longrightarrow\ &(p-1)e^p - (p+1)e^{-p} = 0 \\ \Longrightarrow\ &e^{2p} - \tfrac{p+1}{p-1} = 0 \\ \Longrightarrow\ &e^{2p} - \tfrac{p-1+2}{p-1} = 0 \\ \Longrightarrow\ &e^{2p} - \tfrac{2}{p-1} - 1 = 0.\end{aligned}$$

(b) The equation is not analytically solvable. So, we solve numerically. The N-R iteration is

$$x_{n+1} = x_n - \frac{e^{2x_n} - 2(x_n-1)^{-1} - 1}{2e^{2x_n} + 2(x_n-1)^{-2}}.$$

Running this with $x_0 = 2$, we get $x_1 = 1.5359...$, and then $x_n \to 1.1996....$ So, to 3sf, $p = 1.20$.

4109. Using a double-angle formula, the curve may be expressed as $y = \frac{1}{2}(1 - \cos 2x)$. This has period π, and is tangent to the x axis at $x = 0$:



The area of each region enclosed is

$$\begin{aligned}&\int_0^\pi \sin^2 x\,dx \\ =&\int_0^\pi \tfrac{1}{2}(1 - \cos 2x)\,dx \\ =&\left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^\pi \\ =&\ \tfrac{\pi}{2}, \text{ as required.}\end{aligned}$$

4110. The graph $y = \mathrm{f}(x)$ is symmetrical in the line $x = \frac{k}{2}$. So, the curve must be stationary at $x = \frac{k}{2}$:

$$\begin{aligned}&\ln\tfrac{k}{2} + \ln\left(k - \tfrac{k}{2}\right) = \ln 25 \\ \Longrightarrow\ &2\ln\tfrac{k}{2} = 2\ln 5 \\ \Longrightarrow\ &k = 10.\end{aligned}$$

— Alternative Method —

The derivative is

$$\mathrm{f}'(x) = \frac{1}{x} - \frac{1}{k-x}.$$

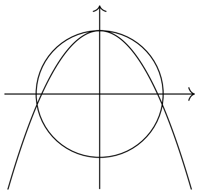
Setting this to zero for SPs,

$$\begin{aligned}&\tfrac{1}{x} - \tfrac{1}{k-x} = 0 \\ \Longrightarrow\ &k - x = x \\ \Longrightarrow\ &x = \tfrac{k}{2}.\end{aligned}$$

The curve is stationary at $x = \frac{k}{2}$, so

$$\begin{aligned}&\ln\tfrac{k}{2} + \ln\left(k - \tfrac{k}{2}\right) = \ln 25 \\ \Longrightarrow\ &2\ln\tfrac{k}{2} = 2\ln 5 \\ \Longrightarrow\ &k = 10.\end{aligned}$$

4111. Both curves are symmetrical in $x = 0$. Hence, the only way of getting an odd number of intersections is if one of these is at $x = 0$. So, we require $a = 1$. The scenario is



Solving for intersections,

$$\begin{aligned} & x^2 + \left(1 - bx^2\right)^2 = 1 \\ \Longrightarrow\ & b^2x^4 + (1-2b)x^2 = 0 \\ \Longrightarrow\ & x^2(b^2x^2 + 1 - 2b) = 0 \\ \Longrightarrow\ & x = 0 \text{ or } x = \pm\sqrt{\frac{2b-1}{b^2}}. \end{aligned}$$

For three distinct points of intersection, the input to the square root (radicand) must be positive:

$$\begin{aligned} & \frac{2b-1}{b^2} > 0 \\ \Longrightarrow\ & 2b - 1 > 0 \\ \Longrightarrow\ & b > \tfrac{1}{2}. \end{aligned}$$

So, $a = 1$ and $b \in \left(\frac{1}{2}, \infty\right)$.

4112. Since the GP is increasing and the common ratio $r > 1$, all terms must be positive. We can express them as a, ar, ar^2, ar^3, ar^4.

- Consider $b + d - 2c$. This is

$$\begin{aligned} & ar + ar^3 - 2ar^2 \\ \equiv\ & ar(r-1)^2. \end{aligned}$$

  All three factors $a, r, (r-1)^2$ are positive. So, $b + d > 2c$.

- Consider $a + e - (b + d)$. This is

$$\begin{aligned} & a + ar^4 - ar - ar^3 \\ \equiv\ & a(r-1)^2(r^2 + r + 1). \end{aligned}$$

  The quadratic factor has $\Delta = -3 < 0$, so it is always positive. Therefore, $a + e > b + d$.

Combining these, $a + e > b + d > 2c$. QED.

4113. For intersections $\sqrt{x} - mx = 0$, which gives $x = 0$ or $x = m^{-2}$. Setting up the relevant integral,

$$\begin{aligned} & \int_0^{m^{-2}} \sqrt{x} - mx\,dx \\ \equiv\ & \left[\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{2}mx^2\right]_0^{m^{-2}} \\ \equiv\ & \tfrac{2}{3}\left(m^{-2}\right)^{\frac{3}{2}} - \tfrac{1}{2}m\left(m^{-2}\right)^2 \\ \equiv\ & \tfrac{2}{3}m^{-3} - \tfrac{1}{2}m^{-3} \\ \equiv\ & \tfrac{1}{6}m^{-3}. \end{aligned}$$

Equating this to 36 gives $m = \frac{1}{6}$.

4114. Sampling without replacement, we multiply the probability of AABB in that order by the number of orders of AABB.

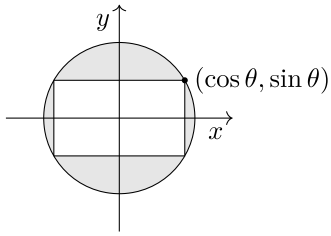
$$p = {}^4\mathrm{C}_2 \times \tfrac{10}{100} \times \tfrac{9}{99} \times \tfrac{90}{98} \times \tfrac{89}{97} = 0.0460 \text{ (3sf)}.$$

4115. (a) Linear transformations such as this preserve features such as points of inflection. The new coordinates are $(a, 2b + 3)$.

(b) Nonlinear transformation such as this don't, in general, preserve features such as points of inflection. Consider $y = \mathrm{h}(x) = x^3$, which has a point of inflection at the origin. The graph $y = (\mathrm{h}(x))^2 = x^6$ is a positive sextic with a local minimum at the origin.

(c) This is a linear transformation. The output is maintained as $y = b$. The input value p which produces this must satisfy $2p + 3 = a$. So, the new coordinates of the point of inflection are $\left(\frac{1}{2}(a-3), b\right)$.

4116. Without loss of generality, let the circle have radius 1, and let the sides of the rectangle be parallel to x and y axes. The vertex in the positive quadrant has coordinates $(\cos\theta, \sin\theta)$.



The area of the rectangle is given by

$$A = 4\cos\theta\sin\theta \equiv 2\sin 2\theta.$$

This is maximised at $2\theta = 90°$, so $\theta = 45°$. This value produces a square. □

— Alternative Method —

Set the problem up as above. The rectangles at θ and $90° - \theta$ are reflections of one another in the line $y = x$, i.e. in $\theta = 45°$. Hence, they have the same area. Since $\theta = 0°$ and $\theta = 90°$ are minima of area, this symmetry dictates that $\theta = 45°$ is a maximum of area. This value produces a square. □

4117. We integrate by parts. Let $u = \ln x$ and $\frac{dv}{dx} = x^2$. This gives $\frac{du}{dx} = \frac{1}{x}$ and $v = \frac{1}{3}x^3$. The integration by parts formula tells us that

$$\begin{aligned} & \int x^2 \ln x\,dx \\ =\ & \tfrac{1}{3}x^3\ln x - \int \tfrac{1}{x}\cdot\tfrac{1}{3}x^3\,dx \\ \equiv\ & \tfrac{1}{3}x^3\ln x - \int \tfrac{1}{3}x^2\,dx \\ =\ & \tfrac{1}{3}x^3\ln x - \tfrac{1}{9}x^3 + c. \end{aligned}$$

4118. On the unit circle, $x^2 + y^2 = 1$. In the positive quadrant, where $x, y \geq 0$, we can rewrite this as $x = \sqrt{1-y^2}$ and $y = \sqrt{1-x^2}$. Substituting these into the LHS of the relationship,

$$\begin{aligned}&x\sqrt{1-y^2} + y\sqrt{1-x^2}\\ &= x \cdot x + y \cdot y\\ &\equiv x^2 + y^2\\ &= 1.\end{aligned}$$

So, points on the circle in the positive quadrant satisfy the relationship. In the negative quadrant, however, where both x and y are negative, then both terms on the LHS $x\sqrt{1-y^2} + y\sqrt{1-x^2}$ must be negative. This precludes their sum equalling 1. Hence, the locus consists of part, but not all, of the unit circle. □

4119. Both sides have a factor of $(x^{\frac{1}{2}} + 3)$:

$$x(x^{\frac{1}{2}} + 3) = x^{\frac{1}{2}} + 3.$$

Since $x^{\frac{1}{2}}$ is positive, this factor cannot be zero, so we can divide through by it. This leaves the solution $x = 1$.

—— Alternative Method ——

Let $u = x^{\frac{1}{2}}$. A polynomial solver gives

$$\begin{aligned}&u^3 + 3u^2 - u - 3 = 0\\ \Longrightarrow\ &u = -3, \pm 1.\end{aligned}$$

The negative u values are not in the range of the square root function, so they produce no x values. Only $u = 1$ does, giving $x = 1$.

4120. (a) Using the quotient rule to look for SPs,

$$\frac{-2x(1+x^2) - (1-x^2)2x}{(1+x^2)^2} = 0 \implies x = 0.$$

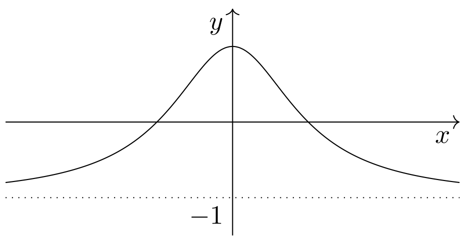
So, there is one stationary point at $(0, 1)$.

(b) Dividing top and bottom by x^2, the curve is

$$y = \frac{\frac{1}{x^2} - 1}{\frac{1}{x^2} + 1}.$$

As $x \to \pm\infty$, the inlaid fractions tend to zero, so $y \to -1$.

(c) The line $y = -1$ is an asymptote. The curve has no vertical asymptotes, as $1 + x^2 > 0$ for all x. The x intercepts are ± 1. Putting this together, the graph is



4121. (a) We notate the probability density function of the normal distribution as

$$\varphi : z \longmapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}.$$

The relevant values are

z	$\varphi(z)$
0	0.39894
0.25	0.38667
0.5	0.35207
0.75	0.30114
1	0.24197.

The trapezium rule formula, with strip width $h = 0.25$, gives

$$\begin{aligned}&\mathbb{P}(0 < Z < 1)\\ \approx\ &\tfrac{1}{8}\Big(0.39894 + 2(0.38667 + 0.35207\\ &\qquad + 0.30114) + 0.24197\Big)\\ =\ &0.3401 \text{ (4sf)}.\end{aligned}$$

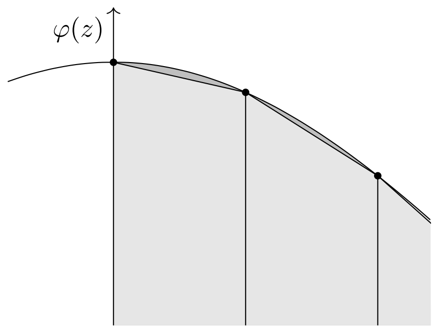
(b) Using a calculator,

$$\mathbb{P}(0 < Z < 1) = 0.3413 \text{ (4sf)}.$$

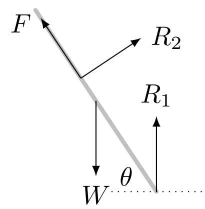
The trapezium rule underestimates this:

$$0.3401 < 0.3413.$$

The standard normal bell curve has points of inflection at $z = \pm 1$, and is concave between them. Hence, the chords for the trapezia all lie below the curve, giving an underestimate for the area/probability. The error is the darker shading below.



4122. (a) The force diagram is



The sloped distance from the ground to the top of the wall is $l \operatorname{cosec}\theta$. Taking moments around the top of the wall,

$$\begin{aligned}&R_1 l \operatorname{cosec}\theta \cos\theta = W(l \operatorname{cosec}\theta - l)\cos\theta\\ \Longrightarrow\ &R_1 \operatorname{cosec}\theta = W(\operatorname{cosec}\theta - 1)\\ \Longrightarrow\ &R_1 = W(1 - \sin\theta).\end{aligned}$$

(b) Taking moments around the base of the ladder,

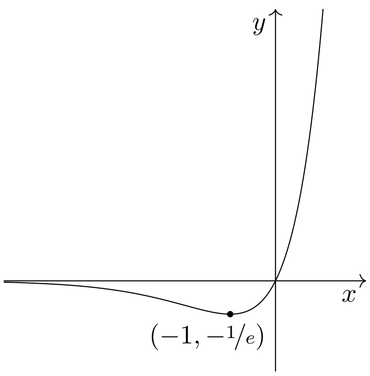
$$\begin{aligned} R_2 l \operatorname{cosec}\theta &= Wl\cos\theta \\ \Longrightarrow R_2 &= W\sin\theta\cos\theta \\ &\equiv \tfrac{1}{2}W\sin 2\theta. \end{aligned}$$

The range of $\sin 2\theta$ is $[-1, 1]$, so the reaction at the wall satisfies $R_2 \leq \frac{1}{2}W$, as required.

4123. We proceed graphically. Differentiating,

$$\begin{aligned} & y = xe^x \\ \Longrightarrow\ & \frac{dy}{dx} = (x+1)e^x \\ \Longrightarrow\ & \frac{d^2y}{dx^2} = (x+2)e^x. \end{aligned}$$

So, there is a stationary point at $x = -1$, which is a local minimum. Also, the curve passes through the origin. As $x \to -\infty$, $y \to 0$ and as $x \to \infty$, $y \to \infty$. Hence, the curve is as follows:



(a) At $p = -1$, the normal is parallel to the y axis, and does not re-intersect the curve.

(b) For $p < -1$, the gradient of the normal is +ve. Such a positive straight line, no matter how steep its gradient, must re-intersect the curve somewhere in the positive quadrant.

— Nota Bene —

If we want, we can show this last fact explicitly, by considering a normal of the form $y = ax + b$, where $a, b > 0$. At $x = 0$, the y difference is positive:

$$ax + b - xe^x > 0.$$

The derivative of the y difference is $a - (x+1)e^x$. As $x \to \infty$, this rate tends to negative infinity, meaning that the curves are getting closer together increasingly quickly. So, eventually, they must cross, as required.

4124. The area of $\triangle ABC$ may be calculated in two ways:

$$\begin{aligned} & \tfrac{1}{2}cl = \tfrac{1}{2}ab \\ \Longrightarrow\ & c^2l^2 = a^2b^2 \\ \Longrightarrow\ & \frac{c^2}{a^2b^2} = \frac{1}{l^2} \\ \Longrightarrow\ & \frac{a^2+b^2}{a^2b^2} = \frac{1}{l^2} \\ \Longrightarrow\ & \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{l^2}, \text{ as required.} \end{aligned}$$

— Alternative Method —

By the standard Pythagorean theorem,

$$|AD|^2 = b^2 - l^2, \quad \text{and} \quad |BD|^2 = a^2 - l^2.$$

We also know that $|AB| + |BD| = c$. Squaring this and using Pythagoras again,

$$|AD|^2 + 2|AD||BD| + |BD|^2 = a^2 + b^2.$$

Substituting in the earlier results,

$$b^2 - l^2 + 2\sqrt{(b^2-l^2)(a^2-l^2)} + a^2 - l^2 = a^2 + b^2.$$

This simplifies to

$$\begin{aligned} & \sqrt{(a^2-l^2)(b^2-l^2)} = l^2 \\ \Longrightarrow\ & (a^2-l^2)(b^2-l^2) = l^4 \\ \Longrightarrow\ & a^2b^2 - l^2a^2 - l^2b^2 + l^4 = l^4 \\ \Longrightarrow\ & a^2b^2 = (a^2+b^2)l^2 \\ \Longrightarrow\ & \frac{1}{l^2} = \frac{a^2+b^2}{a^2b^2} = \frac{1}{a^2} + \frac{1}{b^2}. \end{aligned}$$

Quod erat demonstrandum.

4125. Firstly, we write

$$\int_1^\infty \frac{\ln x + 1}{x^2}\,dx = \lim_{k\to\infty}\int_1^k \frac{\ln x + 1}{x^2}\,dx.$$

We integrate by parts. Let $u = \ln x + 1$ and $v' = 1/x^2$. Then $u' = 1/x$ and $v = -1/x$. So

$$\begin{aligned} & \int \frac{\ln x + 1}{x^2}\,dx \\ &= -\frac{\ln x + 1}{x} + \int \frac{1}{x^2}\,dx \\ &= -\frac{\ln x + 1}{x} - \frac{1}{x} + c. \end{aligned}$$

Hence, the original limit is

$$\begin{aligned} & \lim_{k\to\infty}\left[-\frac{\ln x + 1}{x} - \frac{1}{x}\right]_1^k \\ &= \lim_{k\to\infty}\left(-\frac{\ln k + 1}{k} - \frac{1}{k} - (-1-1)\right) \\ &= 2 - \lim_{k\to\infty}\frac{\ln k}{k} + \frac{2}{k} \\ &= 2, \text{ as required.} \end{aligned}$$

4126. We solve by separation of variables:

$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2-1}{2y} \\ \implies \int 2y\,dy &= \int x^2 - 1\,dx \\ \implies y^2 &= \tfrac{1}{3}x^3 - x + c. \end{aligned}$$

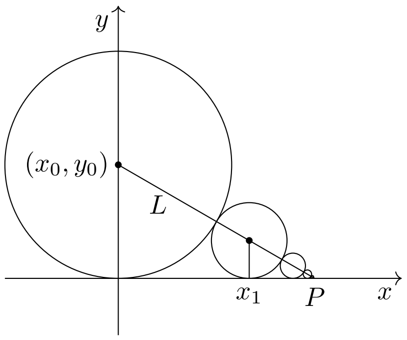
Substituting $(0,0)$ gives $c = 0$. So, the equation of the curve is $3y^2 = x^3 - 3x$. The RHS is zero at $x = 0$ and $x = \pm\sqrt{3}$. Hence, the curve passes through $(\pm\sqrt{3}, 0)$, as well as the origin.

4127. The probability distribution for H is

h	0	1	2	3	4
$\mathbb{P}(H = h)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Since $1 + 6 + 1 = 4 + 4$, $\mathbb{P}(\text{even}) = \mathbb{P}(\text{odd})$.

4128. (a) Consider the straight line L passing through (x_0, y_0) and (x_1, y_1). We do not yet assume that L passes through the other centres. Call its x intercept P.



The triangles formed by P and the vertical radii of C_0 and C_1 are similar. The same is true of the triangles formed by P and the vertical radii of C_1 and C_2. Hence, the centre of C_2 must lie on L. The same is then true for all of the centres.

(b) Let L have angle of inclination θ. Considering the vertical displacement $y_0 - y_1$, $\frac{2}{3} = \frac{4}{3}\sin\theta$, which gives $\theta = 30°$. So, the x intercept of L is $\sqrt{3}$. This is the required limit.

4129. By the product and quotient rules,

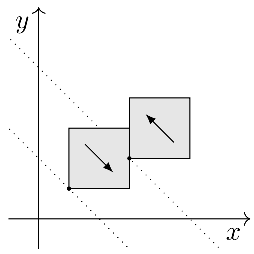
$$\begin{aligned} \mathrm{f}'(x) &= \tan x + x\sec^2 x, \\ \mathrm{f}''(x) &= 2\sec^2 x(1 + x\tan x). \end{aligned}$$

The derivatives of $\mathrm{g}(x) = x^2$ are $\mathrm{g}'(x) = 2x$ and $\mathrm{g}''(x) = 2$. Evaluating both at $x = 0$,

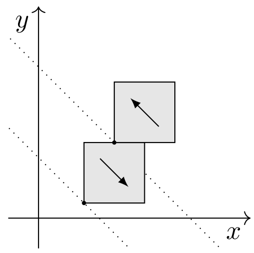
$\mathrm{f}(0) = 0$	$\mathrm{f}'(0) = 0$	$\mathrm{f}''(0) = 2,$
$\mathrm{g}(0) = 0$	$\mathrm{g}'(0) = 0$	$\mathrm{g}''(0) = 2.$

Since the values of the functions and their first and second derivatives agree at $x = 0$, the function $\mathrm{f}(x)$ may be approximated by $\mathrm{g}(x)$ for small x.

4130. (a) In the boundary cases, the edges coincide. By symmetry, the first coincidence is the edges parallel to y at $x = 3$. This occurs at $t = 1$.



Afterwards, the sides parallel to x coincide at $y = 2.5$. This occurs at $t = 1.5$.

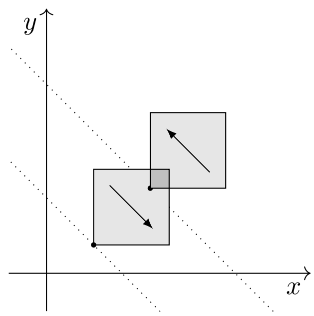


So, the squares overlap for $t \in (1, 1.5)$.

— Nota Bene —

The boundary cases should be excluded here, as an edge in common does not constitute area in common.

(b) By symmetry, the maximal common area must occur at the midpoint of the above interval, i.e. at $t = 1.25$. At this point, the scenario is



At this point, the region in common is a square of side length $1/2$. Hence, the maximal area in common is $1/4$.

4131. Let $u = \ln|\sin x|$ and $\frac{dv}{dx} = \sec^2 x$. Then two (fairly) standard results yield $\frac{du}{dx} = \cot x$ and $v = \tan x$. The parts formula gives

$$\begin{aligned} &\int \sec^2 x \ln|\sin x|\,dx \\ &= \tan x \cdot \ln|\sin x| - \int \cot x \tan x\,dx \\ &= \tan x \cdot \ln|\sin x| - \int 1\,dx \\ &= \tan x \cdot \ln|\sin x| - x + c. \end{aligned}$$

4132. (a) Using the cosine compound-angle formula, with $x = a+b$ and $y = a-b$,

$$\begin{aligned} &\cos x + \cos y \\ =\ &\cos(a+b) + \cos(a-b) \\ \equiv\ &\cos a \cos b - \sin a \sin b \\ &\qquad + \cos a \cos b + \sin a \sin b \\ \equiv\ &2\cos a \cos b. \end{aligned}$$

Solving the original definitions for a and b,

$$a = \frac{x+y}{2} \text{ and } b = \frac{x-y}{2}.$$

Substituting these into the expression above,

$$\cos x + \cos y \equiv 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right).$$

(b) The superposition of two notes is their sum. We can rewrite this using the sum-to-product identity:

$$\begin{aligned} &\cos pt + \cos qt \\ \equiv\ &2\cos\left(\frac{p+q}{2}t\right)\cos\left(\frac{p-q}{2}t\right). \end{aligned}$$

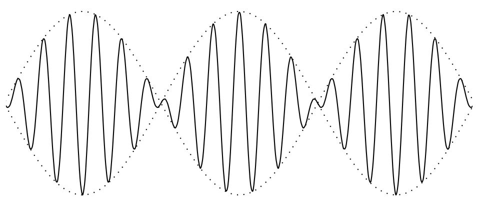
Since p and q differ only by a small amount, the mean of p and q is close to each:

$$\frac{p+q}{2} \approx p, q.$$

But their halved difference is close to zero:

$$\frac{p-q}{2} \approx 0.$$

So, the factor $\cos \frac{p-q}{2}$ represents a wave with a much lower frequency than the original notes. And it scales the outputs of the original note $\cos \frac{p+q}{2}$, causing its volume to undulate slowly. This is the phenomenon of a *beat* in tuning.



4133. The total shaded area is given by

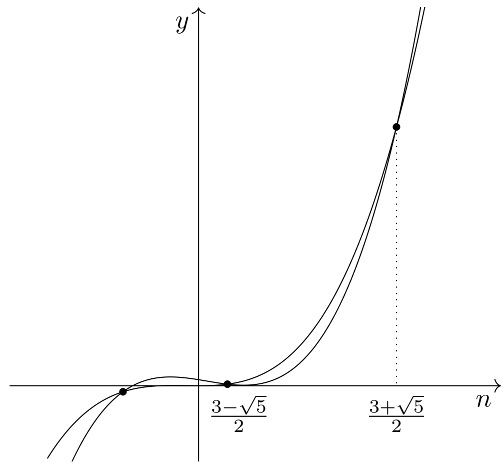
$$\begin{aligned} &\lim_{k\to\infty} \int_2^k (x-1)^{-2}\,dx \\ =\ &\lim_{k\to\infty} \Big[-(x-1)^{-1}\Big]_2^k \\ =\ &\lim_{k\to\infty} \frac{1}{1-k} + 1 \\ =\ &1. \end{aligned}$$

The derivative is $-2(x-1)^{-3}$. At the marked point $(2,1)$, the gradient is -2. So, the triangle below the tangent line has height 1 and base $1/2$: its area is $1/4$. Hence, the tangent line splits the shaded area $1/4 : 3/4$, which is $1:3$, as required.

4134. Consider $y = U_n$ and $y = V_n$ as continuous graphs defined over $\mathbb{R}$. Solving for intersections,

$$\begin{aligned} &3n^3 - n^2 - 2n + 1 = 2n^3 + n^2 \\ \Longrightarrow\ &n^3 - 2n^2 - 2n + 1 = 0 \\ \Longrightarrow\ &n = -1, \tfrac{1}{2}(3 \pm \sqrt{5}). \end{aligned}$$

Since the leading coefficient of U_n is greater than that of V_n, the graph $y = U_n$ is above that of $y = V_n$ for large n.



The relevant surds, to 4dp, are

$$\begin{aligned} n &= \tfrac{3-\sqrt{5}}{2} = 0.3820, \\ n &= \tfrac{3+\sqrt{5}}{2} = 2.6180. \end{aligned}$$

There are two integers in $(0.3820, 2.6180)$.

4135. (a) True. Every cubic equation has a real root, so every cubic has a linear factor. Dividing by this linear factor leaves a quadratic factor.

(b) False. The quartic $x^4 + 1$ is a counterexample: it is irreducible over the reals.

(c) True. Every quintic equation has a real root, so every quintic has a linear factor. Dividing by this linear factor leaves a quartic factor.

4136. Substituting the latter into the former,

$$\begin{aligned} &\log_{10} x + 2\log_{10}(2x-3) = 1 \\ \Longrightarrow\ &\log_{10} x + \log_{10}(2x-3)^2 = 1 \\ \Longrightarrow\ &\log_{10} x(2x-3)^2 = 1 \\ \Longrightarrow\ &x(2x-3)^2 = 10 \\ \Longrightarrow\ &4x^3 - 12x^2 + 9x - 10 = 0. \end{aligned}$$

This has a root at $x = 5/2$. Taking out a factor of $(2x-5)$, we have $(2x-5)(2x^2 - x + 2) = 0$. The quadratic factor has $\Delta = -15 < 0$, so there is a maximum of one (x, y) solution.

At $x = 5/2$, $y = 2$. These satisfy both equations. Hence, the simultaneous equations have exactly one (x, y) solution, as required.

4137. By the product rule,

$$\begin{aligned} y &= x^3 e^x \\ \implies \frac{dy}{dx} &= 3x^2e^x + x^3e^x \\ &= \big(3x^2 + x^3\big)e^x \\ \implies \frac{d^2y}{dx^2} &\equiv \big(6x + 3x^2\big)e^x + (3x^2 + x^3)e^x \\ &\equiv \big(6x + 6x^2 + x^3\big)e^x. \end{aligned}$$

Setting the first derivative to zero,

$$\begin{aligned} 3x^2 + x^3 &= 0 \\ \implies x &= -3, 0. \end{aligned}$$

(a) At $x = -3$, the second derivative is $9e^{-3} > 0$, so there is a local minimum at $x = -3$.

(b) At $x = 0$, the second derivative is 0. We need to show that there is a sign change at $x = 0$. Factorising, the second derivative is

$$\frac{d^2y}{dx^2} = x\big(6 + 6x + x^2\big)e^x.$$

The quadratic factor has a non-zero constant term, so does not change sign at $x = 0$. The exponential factor is always positive. Hence, the single factor of x ensures a sign change in the second derivative. So, there is a point of inflection at $x = 0$, as required.

4138. Solving for intersections,

$$\begin{aligned} x(x + k) + x^2 &= 2 \\ \implies 2x^2 + kx - 2 &= 0 \\ \implies x &= \frac{-k \pm \sqrt{k^2 + 16}}{4}. \end{aligned}$$

The difference between the two x values is

$$x_2 - x_1 = \frac{\sqrt{k^2 + 16}}{2}.$$

Since $y = x + k$ has gradient 1, this gives $|PQ|$ as

$$\sqrt{2} \times \frac{\sqrt{k^2 + 16}}{2}.$$

Equating this to the given value,

$$\begin{aligned} \sqrt{2} \times \frac{\sqrt{k^2 + 16}}{2} &= \frac{5\sqrt{2}}{2} \\ \implies k^2 + 16 &= 25 \\ \implies k &= \pm 3. \end{aligned}$$

4139. Using the conditional probability formula,

$$\begin{aligned} &\mathbb{P}(n \text{ tails} \mid \text{at least } n-1 \text{ tails}) \\ &= \frac{\mathbb{P}(n \text{ tails})}{\mathbb{P}(n-1 \text{ or } n \text{ tails})} \\ &= \frac{\big(\frac{1}{2}\big)^n}{{}^n\mathrm{C}_1\big(\frac{1}{2}\big)^n + \big(\frac{1}{2}\big)^n} \\ &\equiv \frac{1}{n+1}. \end{aligned}$$

— Alternative Method —

The restricted possibility space consists of $n + 1$ outcomes: the n orders of HTT...T and the only order of TT...TT. Of these, the latter is successful. So, the probability is $1/n+1$.

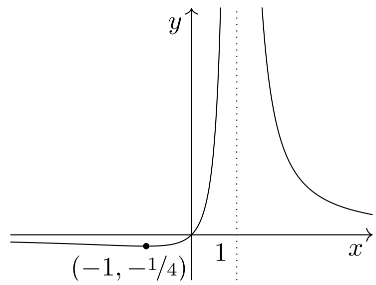
4140. Over a common denominator, the function is

$$\mathrm{f}(x) = \frac{x}{(x-1)^2}.$$

This has a double asymptote at $x = 1$. So, as $x \to 1^{\pm}$, $y \to +\infty$. The degree of the denominator is greater than that of the numerator, so the x axis is horizontal asymptote. For SPs,

$$\begin{aligned} \mathrm{f}'(x) &= \frac{(x-1)^2 - 2x(x-1)}{(x-1)^4} = 0 \\ \implies x &= \pm 1. \end{aligned}$$

The function is undefined at $x = 1$, and has value $-1/4$ at $x = -1$. This must be a minimum. On the domain $[-1, 1)$, therefore, the function achieves its full range, which is $[-1/4, \infty)$.



4141. (a) For SPs, $3x^2 - 3k = 0$. There are three cases:

- $k < 0$: there are no SPs,
- $k = 0$: there is one SP at $(0, 2)$,
- $k > 0$: there are two SPs, at

$$\big(-k^{\frac{1}{2}}, 2k^{\frac{3}{2}} + 2\big) \text{ and } \big(k^{\frac{1}{2}}, -2k^{\frac{3}{2}} + 2\big).$$

(b) We are told that the curve has one x intercept. This is satisfied in the first two cases $k \le 0$. It may also be satisfied in the third case, if the y coordinate of the right-hand SP is greater than zero. Solving $-2k^{\frac{3}{2}} + 2 > 0$, we get $k < 1$. So, $k \in (-\infty, 1)$.

4142. We are told that $\mathrm{f}(a) = \mathrm{f}'(a) = 0$. So, $x = a$ is a root and a stationary point of $\mathrm{f}(x)$. Hence, $x = a$ is a repeated root. By the factor theorem, $(x - a)^2$ is a factor of $\mathrm{f}(x)$.

The same is true of $x = b$.

And we are told that $a \neq b$, so $\mathrm{f}(x)$ has a factor of $(x - a)^2(x - b)^2$, which is quartic. Therefore, $\mathrm{f}(x)$ must have degree at least four. □

4143. The sum of the interior angles is $(n-2)\pi$. So, the mean of the AP is $\frac{n-2}{n}\pi$.

Consider the largest possible common difference, which corresponds to the angles being as distant from the mean as possible. The n-gon is convex, so all interior angles are in $(0,\pi)$. Also, P has at least 5 sides, so the mean angle $\frac{n-2}{n}\pi$ is closer to π than to 0. Hence, the (unattainable) upper bound for the common difference occurs when the greatest angle is π.

Assume the largest interior angle is π. Between this and the mean, the difference is $\pi - \frac{n-2}{n}\pi$, which simplifies to $\frac{2\pi}{n}$. Since the angles form an AP, the smallest and largest angles are equidistant from the mean, so the smallest angle is $\frac{n-2}{n}\pi - \frac{2\pi}{n}$, which simplifies to $\frac{n-4}{n}\pi$. Hence, all interior angles must satisfy

$$\theta > \frac{n-4}{n}\pi.$$

4144. (a) The vertical asymptote is at $x^3 + e^x = 0$. This is not analytically solvable. So, we use the Newton-Raphson method. The iteration, with the function $\mathrm{f}(x) = x^3 + e^x$, is

$$x_{n+1} = x_n - \frac{x_n^3 + e^{x_n}}{3x_n^2 + e^{x_n}}.$$

Running this with $x_0 = 0$, we get $x_1 = -1$, then $x_n \to -0.77288...$ To determine the value to 4sf, we test errors bounds:

$$\mathrm{f}(-0.77295) = -0.00015... < 0,$$
$$\mathrm{f}(-0.77285) = 0.000074... > 0.$$

The sign changes, so $p \in (-0.77295, -0.77285)$. Hence, $p = -0.7729$ (4sf).

(b) Differentiating by the quotient rule,

$$y = \frac{e^x}{x^3 + e^x}$$
$$\Longrightarrow \frac{dy}{dx} = \frac{e^x(x^3+e^x) - e^x(3x^2+e^x)}{(x^3+e^x)^2}.$$

For SPs, the numerator is zero. Since $e^x > 0$ for all x, we can divide through by it, leaving

$$x^3 + e^x - (3x^2 + e^x) = 0$$
$$\Longrightarrow x^3 - 3x^2 = 0$$
$$\Longrightarrow x^2(x-3) = 0.$$

Setting aside $x = 0$ (the point of inflection on the y axis), the exact coordinates of A are

$$\left(3, \frac{e^3}{27+e^3}\right).$$

4145. Using the result $\frac{d}{dx}\sec^2 x = 2\sec^2 x \tan x$, the derivatives are as follows:

$$y = \tan\left(x^2\right)$$
$$\Longrightarrow \frac{dy}{dx} = 2x\sec^2\left(x^2\right)$$
$$\Longrightarrow \frac{d^2y}{dx^2} = 2\sec^2\left(x^2\right) + 8x^2\sec^2\left(x^2\right)\tan\left(x^2\right).$$

Consider the sign of the second derivative. Since $\sec^2\left(x^2\right)$ is positive, we need only consider the sign of $\tan\left(x^2\right)$. On the domain $(0,1)$, the range of $\tan\left(x^2\right)$ is the same as the range of $\tan x$, which is

$$(0, \tan 1 \approx 1.56).$$

Hence, the second derivative is positive on $(0,1)$, which means the function $x \mapsto \tan\left(x^2\right)$ is convex. So, any chord to $y = \tan\left(x^2\right)$ lies above the curve. The trapezium rule will therefore overestimate the value of the integral, irrespective of the number of strips used. □

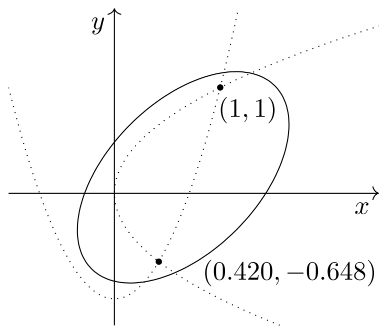
4146. Solving for intersections, $x = (2x^2-1)^2$, which gives $x = 1$ or $x = 0.419643$. So, there are points of intersection at $(1,1)$ and $(0.419643, -0.647799)$. Consider the LHS as a function:

$$\mathrm{f}(x,y) = \left(x + y - \tfrac{4}{5}\right)^2 + 3\left(y - x + \tfrac{1}{2}\right)^2.$$

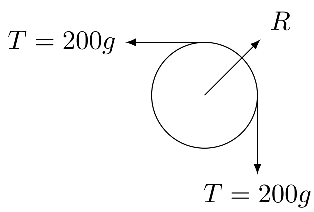
Testing the two points of intersection,

$$\mathrm{f}(1,1) = 2.19 < 3$$
$$\mathrm{f}(0.419643, -0.647799) = 2.02 < 3.$$

Since both of these values are less than 3, both points of intersection lie within the given ellipse.



4147. (a) i. With the load in equilibrium, the tension in the cable is $200g$. So, the force diagram on the right-hand pulley (modelled as light) is



The resultant of the tensions, i.e. the total force exerted by the cable, has magnitude $200\sqrt{2}g \approx 2772$ N. This acts at 45° below the horizontal, to the left. The force on the other pulley is symmetrical, acting down and to the right.

ii. NII for the load gives $T - 200g = 200 \cdot 0.1$, so $T = 1980$. Combining the forces on both pulleys, the horizontal forces cancel. The combined downwards force on the pulleys is $1980 \times 2 = 3960$ N. Since the pulleys are light, this is the same as the combined downwards force on the hoist arm.

(b) The maximum tension in the cable is 10 kN. In this boundary case, NII for the load is

$$\begin{aligned} 10000 - 800g &= 800a \\ \implies a &= 2.7 \end{aligned}$$

So, the maximum acceleration is 2.7 ms^{-2}. This is upwards. Downwards, any acceleration is safe regarding the force on the hoist arm.

4148. Rotation by 180° about the origin is the same as reflection in both x and y axes. So, we replace x by $-x$ and y by $-y$, giving the equation of the transformed graph as $\mathrm{f}_1(-x)\,\mathrm{f}_2(-y) = 1$.

4149. The inequality describes the interior of an ellipse. Its boundary equation is $a^2 + 3b^2 = 10$. Solving for intersections with the parabola,

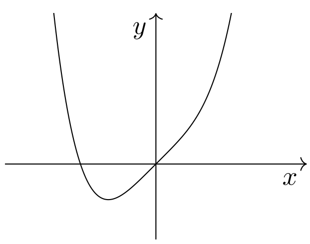
$$\begin{aligned} 12 - b + 3b^2 &= 10 \\ \implies 3b^2 - b + 2 &= 0. \end{aligned}$$

This has $\Delta = -23 < 0$. So, the parabola does not intersect the boundary ellipse. The parabola must therefore lie outside the ellipse everywhere. Hence, there are no points which simultaneously satisfy the inequality and the equation.

4150. Differentiating the given curve twice,

$$\begin{aligned} y &= x^4 + x \\ \implies \frac{dy}{dx} &= 4x^3 + 1 \\ \implies \frac{d^2y}{dx^2} &= 12x^2. \end{aligned}$$

The second derivative is zero at $x = 0$, but the root is a double root. So, the second derivative has the same sign (positive) for $x < 0$ and $x > 0$. Hence, the origin is not a point of inflection.



4151. (a) Substituting into the LHS,

$$\begin{aligned} (xy-1)^2 &= \big(\operatorname{cosec} t(\sin t + \cos t) - 1\big)^2 + 1 \\ &\equiv \cot^2 t + 1 \\ &\equiv \operatorname{cosec}^2 t \\ &= x^2. \end{aligned}$$

(b) Setting $\frac{dx}{dt} = 0$, we get $-\operatorname{cosec} t \cot t = 0$. The cosec function is never zero, so $\cot t = 0$. The relevant roots are $t = \pm\pi/2$. Substituting these values in, the points are at $A : (-1, -1)$ and $B : (1, 1)$.

4152. Taking out a factor of $1/x^4$ from the right-hand bracket, we have

$$\begin{aligned} & (x-1)^4 \left(1 + \tfrac{1}{x}\right)^4 \\ \equiv\; & \tfrac{1}{x^4}(x-1)^4(x+1)^4 \\ \equiv\; & \tfrac{1}{x^4}(x^2-1)^4 \\ \equiv\; & \tfrac{1}{x^4}(x^8 - 4x^6 + 6x^4 - 4x^2 + 1) \\ \equiv\; & x^4 - 4x^2 + 6 - 4x^{-2} + x^{-4}. \end{aligned}$$

The constant term is 6.

4153. The tangent at $x = a$ is $y = 2ax - a^2$. At $x = a - 3$, it is $y = 2(a-3)x - (a-3)^2$. Each of these passes through $(b, -2)$. So, we have equations

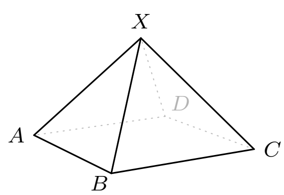
$$\begin{aligned} -2 &= 2ab - a^2, \\ -2 &= 2(a-3)b - (a-3)^2. \end{aligned}$$

The former gives $b = \frac{a^2-2}{2a}$. Substituting this into the latter,

$$\begin{aligned} -2 &= 2(a-3)\tfrac{a^2-2}{2a} - (a-3)^2 \\ \implies -4a &= 2(a-3)(a^2-2) - 2a(a^2 - 6a + 9) \\ \implies a^2 &- 3a^2 + 2 = 0 \\ \implies a &= 1, 2. \end{aligned}$$

So, $a = 1$, $b = -1/2$ or $a = 2$, $b = 1/2$.

4154. The centres of the four spheres A, B, C, D and the point from which the strings are suspended X form a square based pyramid, in which all edges lengths are $2r$.



The four tensions act along edges AX, BX, CX, DX. The angle of inclination of these edges, above the face $ABCD$, is 45°. The forces on each sphere are tension, weight, and two horizontal reaction forces from the neighbouring spheres. The only vertical forces are tension and weight. Resolving vertically, $T \sin 45° - mg = 0$, so $T = \sqrt{2}mg$. □

4155. Firstly, consider $x^2 + y^2$. This is

$$\begin{aligned}&4(1-\cos t)^2\cos^2 t + 4(1-\cos t)^2\sin^2 t\\ \equiv\ &4(1-\cos t)^2(\cos^2 t + \sin^2 t)\\ \equiv\ &4(1-\cos t)^2.\end{aligned}$$

Each term of the LHS of the Cartesian equation contains a factor of $4(1-\cos t)^2$. The first term is

$$(x^2+y^2)^2 = 4(1-\cos t)^2(4 - 8\cos t + 4\cos^2 t).$$

The second term is

$$4x\big(x^2+y^2\big) = 4(1-\cos t)^2(8\cos t - 8\cos^2 t).$$

The third term is

$$-4y^2 = 4(1-\cos t)^2(-4\sin^2 t).$$

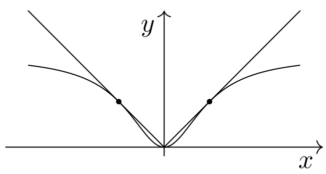
Adding these together, the LHS is

$$\begin{aligned}&4(1-\cos t)^2\big(4 - 4\cos^2 t - 4\sin^2 t\big)\\ \equiv\ &4(1-\cos t)^2\big(4 - 4(\cos^2 t + \sin^2 t)\big)\\ \equiv\ &0, \text{ as required.}\end{aligned}$$

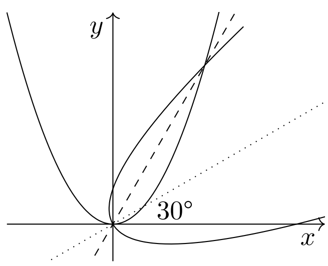
4156. Both the LHS and the RHS have even symmetry. So, we need only show the result for non-negative x. Since $1+x^2$ is positive, we can multiply through, while maintaining the direction of the inequality:

$$\begin{aligned}&\frac{2x^2}{1+x^2} \le x\\ \iff\ &2x^2 \le x + x^3\\ \iff\ &0 \le x(x-1)^2.\end{aligned}$$

For $x \ge 0$, this last inequality holds. Symmetry then gives the full result. The graphs of $y = $ LHS and $y = $ RHS are shown below.



4157. The y axis is the line of symmetry of $y = x^2$. The angle between this and the line $y = \sqrt{3}x$ is $90° - \arctan\sqrt{3}$, which is 30°. When reflected, the image of the y axis is therefore 30° below the line $y = \sqrt{3}x$. It is 30° above the x axis.



At inclination 30°, the gradient is $\tan 30° = 1/\sqrt{3}$. So, the new line of symmetry is $y = \frac{x}{\sqrt{3}}$.

4158. We know that $\frac{dx}{dt} = x^2$. Separating the variables,

$$\begin{aligned}&\frac{dx}{dt} = x^2\\ \Longrightarrow\ &\int x^{-2}\,dx = \int 1\,dt\\ \Longrightarrow\ &-x^{-1} = t + c\\ \Longrightarrow\ &x = \frac{1}{-c-t}.\end{aligned}$$

Substituting $t = 0$, $x = 1$, we get $c = -1$, so the position is given by

$$x = \frac{1}{1-t}.$$

This has an asymptote at $t = 1$, approaching which the position x grows without bound. At this point, and therefore for all $t \ge 1$, the model breaks down. It can only apply during the first second of motion.

4159. The third derivative of f is positive everywhere. Hence, it must be a polynomial of even degree. Integrating three times switches the parity three times, so f must have odd degree. The range of any polynomial of odd degree is $\mathbb{R}$. □

4160. Using the first Pythagorean trig identity,

$$\begin{aligned}&\sin^3 x - 8\cos^2 x + 21\sin x + 26 = 0\\ \Longrightarrow\ &\sin^3 x - 8(1-\sin^2 x) + 21\sin x + 26 = 0\\ \Longrightarrow\ &\sin^3 x + 8\sin^2 x + 21\sin x + 18 = 0.\end{aligned}$$

This is a cubic in $\sin x$. Let $z = \sin x$, giving $z^3 + 8z^2 + 21z + 18 = 0$. Factorising,

$$\begin{aligned}&z^3 + 8z^2 + 21z + 18 = 0\\ \Longrightarrow\ &(z+2)(z+3)^2 = 0\\ \Longrightarrow\ &z = -2, -3.\end{aligned}$$

So, we have $\sin x = -2, -3$. But the range of the sine function is $[-1, 1]$. Hence, no x values satisfy the original equation, as required.

4161. Differentiating with respect to x,

$$\begin{aligned}&y = x^3 + y^3\\ \Longrightarrow\ &\frac{dy}{dx} = 3x^2 + 3y^2\frac{dy}{dx}\\ \Longrightarrow\ &\frac{dy}{dx} = \frac{3x^2}{1-3y^2}.\end{aligned}$$

At the three points in question, the x coordinate is zero, so $\frac{dy}{dx} = 0$. Furthermore, as x passes through 0, there is no change of sign in the gradient, due to the double factor of x in the numerator. Hence, the three y axis intercepts are (stationary) points of inflection.

— Nota Bene —

At a stationary point, the following conditions for a point of inflection are equivalent:

- the first derivative does not change sign,
- the second derivative is zero and changes sign,
- the second derivative is zero and the third derivative is non-zero.

The first of these conditions does not carry through to non-stationary points of inflection, but the other two do.

4162. (a) $X_1 \sim N\big(\mu, \sigma^2\big)$,

(b) $\bar{X} \sim N\Big(\mu, \frac{\sigma^2}{n}\Big)$,

(c) $X_1 + X_2 + ... + X_k \sim N\big(k\mu, k\sigma^2\big)$,

(d) The distribution of $aX_i + b$ is $N\big(a\mu + b, a^2\sigma^2\big)$. Then, using part (b),

$$\frac{1}{n}\sum_{i=1}^{n}(aX_i + b) \sim N\Big(a\mu + b, \frac{a^2\sigma^2}{n}\Big).$$

4163. Radius OP has gradient $\tan\theta$. So, tangent AB has gradient $-\cot\theta$. Its equation is

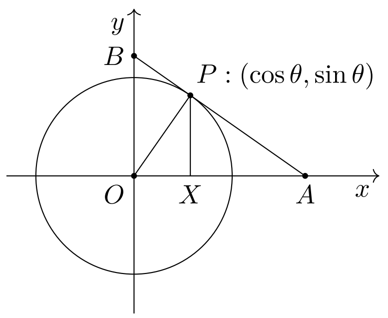
$$y - \sin\theta = -\cot\theta(x - \cos\theta).$$

The axis intercepts are therefore $A : (\sec\theta, 0)$ and $B : (0, \operatorname{cosec}\theta)$. The area of $\triangle OAB$ is

$$\begin{aligned} A_{\triangle} &= \tfrac{1}{2}\sec\theta\operatorname{cosec}\theta \\ &\equiv \frac{1}{2\cos\theta\sin\theta} \\ &\equiv \frac{1}{\sin 2\theta} \\ &\equiv \operatorname{cosec} 2\theta, \text{ as required.} \end{aligned}$$

——— Alternative Method ———

Triangles OAP, OBP and OXP are all similar.



$|OX| = \cos\theta$ and $|XP| = \sin\theta$. So, scaling up the hypotenuse of OXP,

$$|OA| = |OP| \times \frac{|OP|}{|OX|} = \frac{1}{\cos\theta} \equiv \sec\theta.$$

By symmetry, $|OB| = \operatorname{cosec}\theta$. The rest of the proof follows as in the first solution.

4164. There is a common factor of x on top and bottom. Cancelling this, the integrand is

$$\frac{18x^2 - 6}{3x^2 - 2x - 1}.$$

As a proper algebraic fraction, this is

$$\begin{aligned} &\frac{6(3x^2 - 2x - 1) + 12x}{3x^2 - 2x - 1} \\ \equiv\ & 6 + \frac{12x}{3x^2 - 2x - 1}. \end{aligned}$$

The denominator factorises as $(3x + 1)(x - 1)$. Writing in partial fractions,

$$12x \equiv A(x - 1) + B(3x + 1).$$

Equating coefficients, $A + 3B = 12$ and $B - A = 0$. This gives $A = B = 3$. We can now integrate:

$$\begin{aligned} &\int \frac{18x^3 - 6x}{3x^3 - 2x^2 - x}\,dx \\ =& \int 6 + \frac{3}{3x + 1} + \frac{3}{x - 1}\,dx \\ =&\ 6x + \ln|3x + 1| + 3\ln|x - 1| + c. \end{aligned}$$

——— Nota Bene ———

The relevant polynomial long division is

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad 6 \\ \hline 3x^2 - 2x - 1 & \ 18x^2 \quad\quad\quad - 6 \\ & -18x^2 + 12x + 6 \\ \hline & \quad\quad\quad 12x + 0 \end{array}$$

4165. The equation is

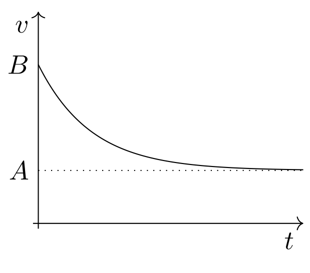
$$\begin{aligned} & 2\sqrt{x} + 1 - \sqrt{2x + 1} = 0 \\ \Longrightarrow\ & 2\sqrt{x} + 1 = \sqrt{2x + 1} \\ \Longrightarrow\ & 4x + 4\sqrt{x} + 1 = 2x + 1 \\ \Longrightarrow\ & 2\sqrt{x} + x = 0 \\ \Longrightarrow\ & \sqrt{x}\big(2 + \sqrt{x}\big) = 0. \end{aligned}$$

The range of $\sqrt{x}$ is $\mathbb{R}^+$, so the latter factor has no real roots. Hence, $x = 0$ is the only real root of the equation.

4166. (a) We rearrange to the form

$$v = A + (B - A)e^{-kt}.$$

In this form, it is clear that the velocity decays towards $v = A$, from initial value $v = B$. The velocity-time graph is



(b) A is terminal velocity, B is initial velocity, and k describes the rate at which the object slows down. It is not acceleration, however, as seen in the answer to (c).

(c) $0 < A < B$, so this model doesn't describe objects falling from rest. It describes objects projected downwards, with initial downward speeds greater than their terminal velocities.

(d) B must also have units ms^{-1}. The inputs into the exponential function are numbers without units, so kt should have no units. Hence, the units of k are s^{-1}.

(e) The displacement is given by the integral of velocity:

$$\begin{aligned} s &= \int A + (B - A)e^{-kt}\, dt \\ &= At - \frac{B-A}{k}e^{-kt} + c. \end{aligned}$$

By definition, $s = 0$ at $t = 0$. Substituting in,

$$\begin{aligned} 0 &= -\frac{B-A}{k} + c \\ \implies c &= \frac{B-A}{k}. \end{aligned}$$

Hence, the displacement is

$$\begin{aligned} s &= At - \frac{B-A}{k}e^{-kt} + \frac{B-A}{k} \\ &\equiv At + \frac{B-A}{k}\left(1 - e^{-kt}\right), \text{ as required.} \end{aligned}$$

4167. The change of base formula is

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

Using this to convert to natural logs,

$$\begin{aligned} \text{f}(x) &= \frac{\ln x}{\ln a} - \frac{\ln x}{\ln b} \\ &\equiv \left(\frac{1}{\ln a} - \frac{1}{\ln b}\right)\ln x. \end{aligned}$$

The ln function is increasing everywhere. So, since $1 < a < b$, we know that $0 < \ln a < \ln b$. Hence,

$$\frac{1}{\ln a} > \frac{1}{\ln b}.$$

So, $\text{f}(x) = k \ln x$, where $k > 0$, and is therefore an increasing function. □

4168. (a) The derivative of $\tan x$ is $\sec^2 x$. At $x = 0$, this has value 1. The gradient of $y = -x$ is -1, so curve and line are normal at the origin.

(b) For intersections, $x + \tan x = 0$. This is not analytically solvable. A fixed-point iteration (for the first positive intersection) is

$$x_{n+1} = \arctan(-x_n) + \pi.$$

Running this with $x_0 = 0.5$, we get $x \approx 2.03$. Confirming with error bounds,

$$\begin{aligned} 2.025 + \tan(2.025) &= -0.023... < 0, \\ 2.035 + \tan(2.035) &= 0.037... > 0. \end{aligned}$$

So, the first intersection lies in $(2.025, 2.035)$. Over this interval, the gradient takes values between $\sec^2(2.025)$ and $\sec^2(2.035)$, which is $m \in (4.99, 5.19)$. Therefore, the curve and the line are not normal to each other at the first point of intersection. Hence, there must be a shorter path between two successive branches than the one lying along $y = -x$.

— Nota Bene —

The addition of π in the fixed-point iteration is necessary because the graph $y = \arctan x$ only contains (a reflection in $y = x$ of) the *central* branch of the graph $y = \tan x$. Using the iteration $x_{n+1} = \arctan(-x_n)$ gives only $x = 0$. The function $x \mapsto \arctan(x) + \pi$ is a non-standard inverse of the tan function, in the same way that $x \mapsto -\sqrt{x}$ is a non-standard inverse of the squaring function.

4169. (a) This is well defined. Since $x \neq 1$, we can cancel factors of $(x - 1)$ from top and bottom. The limit then has value 1.

(b) This is not well defined. The value depends on the direction from which we approach 1. First, cancel factors of $(x + 1)$. Then consider the approach from above, in which $x - 1 > 0$:

$$\lim_{x\to1^+} \frac{|x-1|}{x-1} = \lim_{x\to1^+} \frac{x-1}{x-1} = 1.$$

But approaching from below, $x - 1 < 0$:

$$\lim_{x\to1^-} \frac{|x-1|}{x-1} = \lim_{x\to1^-} \frac{-(x-1)}{x-1} = -1.$$

(c) This is not well defined, for the same reasons as in (b). The extra modulus sign on $|x+1|$ has no impact: anywhere in the vicinity of $x = 1$, $x + 1$ is positive, so the new mod sign has no effect.

4170. Call the cubics $\text{f}(x)$ and $\text{g}(x)$, and consider the equation $\text{f}(x) - \text{g}(x) = 0$. This is a polynomial equation of degree at most three. We are told that it has four distinct roots. But a polynomial of degree $n \geq 1$ can have at most n roots. So, $\text{f}(x) - \text{g}(x) = 0$ must be a polynomial of degree zero, with infinitely many roots, i.e. $\text{f}(x) - \text{g}(x)$ must be identically zero for all $x \in \mathbb{R}$. This means that $y = \text{f}(x)$ and $y = \text{g}(x)$ must be the same cubic graph. □

4171. (a) The cross-sections both have y intercept -32, and roots at $x = \pm 8$. So, their depths are 32 m and their widths are 16 m.

(b) At A, the x coordinates of the edges of the water are given by $\frac{1}{2}x^2 - 32 = -3.27$, so that $x = \pm 7.58$. The cross-sectional area is

$$\int_{-7.58}^{7.58} -3.27 - \left(\tfrac{1}{2}x^2 - 32\right) dx = 290.374.$$

We perform the same calculation at B, calling the x coordinates of the edges of the water $\pm k$. The cross-sectional area is

$$\begin{aligned} &\int_{-k}^{k} \tfrac{1}{128}k^4 - 32 - \left(\tfrac{1}{128}x^4 - 32\right) dx \\ \equiv\ &\left[\tfrac{k^4}{128}x - \tfrac{1}{640}x^5\right]_{-k}^{k} \\ \equiv\ &\tfrac{1}{80}k^5. \end{aligned}$$

Setting this equal to the result at A,

$$\begin{aligned} &\tfrac{1}{80}k^5 = 290.374 \\ \Longrightarrow\ &k = 7.4681. \end{aligned}$$

Substituting back in, the depth of the surface at B is 7.70 metres (3sf) below ground level.

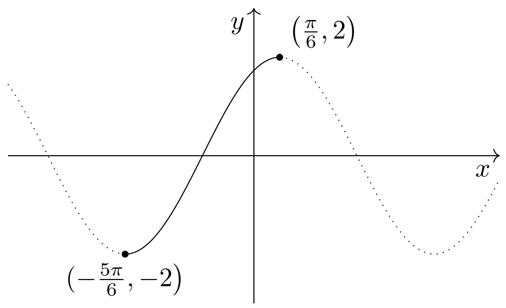
4172. Writing the function in harmonic form,

$$\begin{aligned} \sin x + \sqrt{3}\cos x &\equiv R\sin(x + \alpha) \\ &\equiv R\sin x\cos\alpha + R\sin\alpha\cos x. \end{aligned}$$

Equating coefficients, $R\cos\alpha = 1$, $R\sin\alpha = \sqrt{3}$. So, $R = 2$ and $\alpha = \arctan\sqrt{3} = \frac{\pi}{3}$. This gives

$$\mathrm{f}(x) = 2\sin\left(x + \tfrac{\pi}{3}\right).$$

This is a stretched and translated sine wave. The largest interval containing zero over which it is one-to-one and therefore invertible is shown below:



The interval is $\left[-\frac{5\pi}{6}, \frac{\pi}{6}\right]$.

4173. This is a cubic in e^x. Taking out a factor of e^x,

$$\begin{aligned} &e^{3x} - e^{2x} + 5e^x > 0 \\ \Longleftrightarrow\ &e^x(e^{2x} - e^x + 5) > 0. \end{aligned}$$

Since $e^x > 0$, this is true iff the second factor is positive. And $(e^{2x} - e^x + 5)$ is a positive quadratic in e^x with discriminant $\Delta = -19 < 0$, so it is also positive for all x. Hence, all real x values satisfy the inequality: the set in question is $\mathbb{R}$.

4174. By Pythagoras, $x^2 + y^2 = r^2$. Also (noting reversal of sin and cos due to an angle with the y axis),

$$\begin{aligned} x &= r\sin\theta, \\ y &= r\cos\theta. \end{aligned}$$

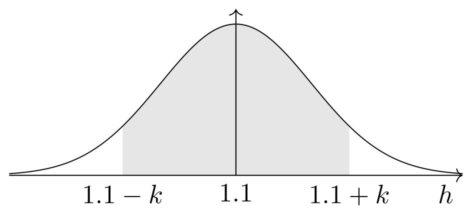
Substituting these in,

$$\begin{aligned} &r = 2(\sec\theta - \cos\theta) \\ \Longrightarrow\ &r = 2\left(\tfrac{r}{y} - \tfrac{y}{r}\right) \\ \Longrightarrow\ &r^2 y = 2\left(r^2 - y^2\right) \\ \Longrightarrow\ &\left(x^2 + y^2\right)y = 2\left(x^2 + y^2 - y^2\right) \\ \Longrightarrow\ &\left(x^2 + y^2\right)y = 2x^2, \text{ as required.} \end{aligned}$$

4175. (a) $\mathbb{P}(H < 1) = 0.309$ (3dp).

(b) We need k such that

$$\mathbb{P}(\mu - k < H < \mu + k) = 0.9.$$



Each unshaded tail has probability 0.05. So, using a standard normal $Z \sim N(0, 1)$, the z value is 1.645. Hence,

$$\frac{(1.1 + k) - 1.1}{0.2} = 1.645 \implies k = 0.329.$$

The required bounds are $a, b = 1.1 \pm 0.329$. The interval is $(0.771, 1.429)$ metres, to 3dp.

4176. Firstly, consider $x > 0$. By definition, $x \equiv e^{\ln x}$. Differentiating this,

$$\begin{aligned} &x \equiv e^{\ln x} \\ \Longrightarrow\ &1 \equiv e^{\ln x} \cdot \tfrac{d}{dx}(\ln x) \\ \Longrightarrow\ &1 \equiv x \cdot \tfrac{d}{dx}(\ln x) \\ \Longrightarrow\ &\tfrac{d}{dx}(\ln x) \equiv \tfrac{1}{x}. \end{aligned}$$

This proves the integral result for $x > 0$:

$$\int \frac{1}{x}\, dx = \ln x + c.$$

Secondly, consider $x < 0$. This time, $x \equiv -e^{\ln(-x)}$. Proceeding as before,

$$\begin{aligned} &x \equiv -e^{\ln(-x)} \\ \Longrightarrow\ &1 \equiv -e^{\ln(-x)} \cdot \tfrac{d}{dx}\big(\ln(-x)\big) \\ \Longrightarrow\ &1 \equiv x \cdot \tfrac{d}{dx}\big(\ln(-x)\big) \\ \Longrightarrow\ &\tfrac{d}{dx}\big(\ln(-x)\big) \equiv \tfrac{1}{x}. \end{aligned}$$

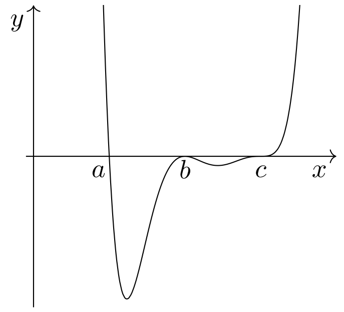
This proves the integral result for $x < 0$:

$$\int \frac{1}{x}\, dx = \ln(-x) + c.$$

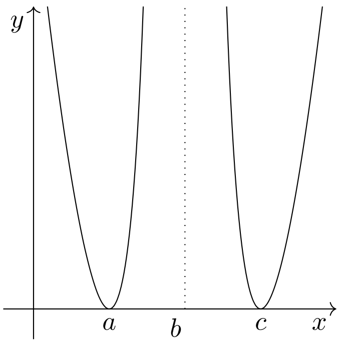
The two results can be expressed together, using a modulus function:

$$\int \frac{1}{x}\,dx = \ln|x| + c, \text{ as required.}$$

4177. (a) This is a positive sextic with a single root at $x = a$, a double root at $x = b$ and a triple root at $x = c$:



(b) The graph has double roots at $x = a$ and $x = c$, and a double asymptote at $x = b$. In its behaviour for large x, it is close to parabolic:

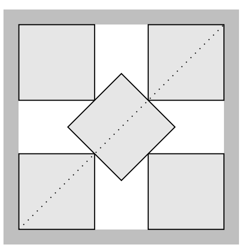


4178. (a) We only have information about the second derivative. To analyse the first derivative, we would need to integrate. This would introduce an unknown constant of integration.

(b) The function g is convex when its second derivative is positive. Since the graph shown is positive for all x values other than -2, the original curve $y = \mathrm{g}(x)$ is convex on $\mathbb{R} \setminus \{-2\}$.

(c) The second derivative is zero at $x = -2$, but does not change sign: g is convex both sides of $x = -2$. So, $x = -2$ is not a point of inflection.

4179. Consider the following configuration of five unit squares, enclosed by a larger square:



The length of the diagonal is $2\sqrt{2} + 1$. Dividing this by $\sqrt{2}$ gives the side length of the enclosing square as $2 + \sqrt{2}/2$, which proves the result.

4180. Differentiating implicitly,

$$\begin{aligned} & x^2y^3 - 2x = 3y \\ \Longrightarrow\ & 2xy^3 + 3x^2y^2\frac{dy}{dx} - 2 = 3\frac{dy}{dx}. \end{aligned}$$

Setting $\dfrac{dy}{dx} = 0$ for stationary points,

$$\begin{aligned} & 2xy^3 - 2 = 0 \\ \Longrightarrow\ & y = x^{-\frac{1}{3}}. \end{aligned}$$

Substituting this into the original relation,

$$\begin{aligned} & x - 2x = 3x^{-\frac{1}{3}} \\ \Longrightarrow\ & x^{\frac{4}{3}} = -3. \end{aligned}$$

Since $x^{\frac{4}{3}} \geq 0$, this has no real roots. Hence, y is never stationary with respect to x, as required.

4181. (a) The boundary equation is

$$\begin{aligned} & a + b = 2\sqrt{ab} \\ \Longrightarrow\ & (a+b)^2 = 4ab \\ \Longrightarrow\ & a^2 + 2ab + b^2 = 4ab \\ \Longrightarrow\ & a^2 - 2ab + b^2 = 0 \\ \Longrightarrow\ & (a-b)^2 = 0 \\ \Longrightarrow\ & a = b. \end{aligned}$$

(b) We can follow the above argument in reverse, starting with the known fact that any square is non-negative:

$$\begin{aligned} & (a-b)^2 \geq 0 \\ \Longrightarrow\ & a^2 - 2ab + b^2 \geq 0 \\ \Longrightarrow\ & a^2 + 2ab + b^2 \geq 4ab \\ \Longrightarrow\ & (a+b)^2 \geq 4ab. \end{aligned}$$

Since $a, b \geq 0$, we can take the square root, which yields the AM-GM inequality:

$$\frac{a+b}{2} \geq 2\sqrt{ab}.$$

Quod erat demonstrandum.

4182. (a) With $k = 1$, the derivative is

$$\mathrm{f}'(x) = -\sin x + \tan x \sec x.$$

Setting this to zero for SPs,

$$\begin{aligned} & -\sin x + \tan x \sec x = 0 \\ \Longrightarrow\ & -\sin x \cos^2 x + \sin x = 0 \\ \Longrightarrow\ & \sin x(\cos^2 x - 1) = 0 \\ \Longrightarrow\ & \sin x = 0 \text{ or } \cos^2 x = \pm 1. \end{aligned}$$

Over the domain $[0, \pi/2)$, this gives only $x = 0$, which is at the boundary. The derivative is non-negative over $[0, \pi/2)$. So, the function is one-to-one over this domain, and is invertible.

(b) With $k = \frac{1}{2}$, the derivative is

$$\mathrm{f}'(x) = -\sin x + \tfrac{1}{2}\tan x \sec x.$$

Setting this to zero for SPs,

$$\begin{aligned} & -\sin x + \tfrac{1}{2}\tan x \sec x = 0 \\ \Longrightarrow\; & -\sin x \cos^2 x + \tfrac{1}{2}\sin x = 0 \\ \Longrightarrow\; & \sin x\big(\cos^2 x - \tfrac{1}{2}\big) = 0. \end{aligned}$$

This has another SP at $x = \pi/4$. Analysing the values of the function at the boundaries of the domain and at the SP:

x	0	$\pi/4$	$\to \pi/2$
$\mathrm{f}(x)$	2	$\sqrt{2}$	$\to \infty.$

Since $\sqrt{2} < 2$, the function is not one-to-one over the domain, and so is not invertible.

4183. For tangents parallel to y,

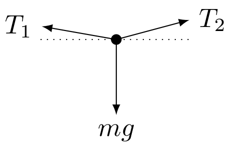
$$\begin{aligned} & \frac{dx}{dt} = 3t^2 - 3 = 0 \\ \Longrightarrow\; & t = \pm 1. \end{aligned}$$

This gives points $A, B : (\pm 2, 1)$. To find P and Q, we set $x = 2$ and solve:

$$\begin{aligned} & t^3 - 3t + 2 = 0 \\ \Longrightarrow\; & (t-1)^2(t+2) = 0 \\ \Longrightarrow\; & t = 1, -2. \end{aligned}$$

At $t = -2$, the coordinates are $(2, 4)$. This is P. By symmetry, Q is $(-2, 4)$. Quadrilateral $ABQP$ has width 4 and height 3, so its area is 12.

4184. (a) The force diagram for the stuntman is



Horizontal equilibrium requires

$$T_2 \cos 15° - T_1 \cos 10° = 0.$$

So, $T_2 \neq T_1$. The tensions differ in the two sections of wire, so the stuntman is exerting a frictional force on the wire, and vice versa. Hence, the contact cannot be smooth.

(b) i. The wire is not symmetrical horizontally, so the perpendicular reaction force exerted cannot be vertical. Hence, it cannot have magnitude mg.

ii. The sections of wire are nearly symmetrical horizontally, so reaction is nearly vertical. Its magnitude, therefore, is $R \approx mg$.

(c) The resultant of the tensions must be directed vertically upwards, with magnitude mg N.

— Nota Bene —

The resultant force in part (c) is a combination of reaction and frictional components. While the reaction is not quite vertical, the friction is not quite horizontal. In combination, they add to mg to counteract the weight of the stuntman.

4185. There is a common factor of $(x+2)^3$. This gives a root at $x = -2$. What remains is quartic:

$$(x-2)^4 + x^4 = 0.$$

Both terms are fourth powers, so non-negative. Hence, their sum can only be equal to zero if both are equal to zero. But this is only true if $x = 0$ and $x = 2$ simultaneously, which is not possible. The original equation, therefore, has exactly one real root $x = -2$.

— Alternative Method —

There is a common factor of $(x+2)^3$. This gives a root at $x = -2$. What remains is quartic:

$$\begin{aligned} & (x-2)^4 + x^4 = 0 \\ \Longrightarrow\; & 2x^4 - 8x^3 + 24x^2 - 32x + 16 = 0 \\ \Longrightarrow\; & x^4 - 4x^3 + 12x^2 - 16x + 8 = 0. \end{aligned}$$

We need to show this has no real roots. Looking for stationary values of the quartic,

$$\begin{aligned} & 4x^3 - 12x^2 + 24x - 16 = 0 \\ \Longrightarrow\; & 4(x-1)(x^2 - 2x + 4) = 0. \end{aligned}$$

The quadratic has $\Delta = -12 < 0$, so no real roots. Hence, the quartic has one SP, at $x = 1$. It is a positive quartic, so this must be a local and global minimum. Evaluating,

$$x^4 - 4x^3 + 12x^2 - 16x + 8\Big|_{x=1} = 1.$$

Since this is positive, the quartic must be positive everywhere. Hence, the original equation has one real root, at $x = -2$.

4186. We need to find

$$y = \int \frac{e^x + 1}{e^x + 2}\, dx.$$

Let $u = e^x + 2$, so that $du = e^x\, dx$. This gives

$$dx = \frac{du}{u - 2}.$$

Enacting the substitution,

$$y = \int \frac{u - 1}{u(u-2)}\, du.$$

Writing the integrand in partial fractions,

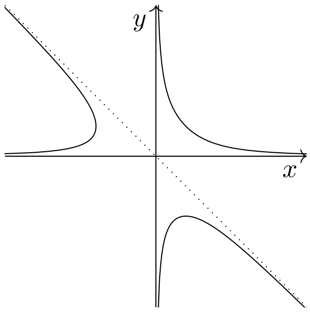
$$\begin{aligned} y &= \tfrac{1}{2}\int \frac{1}{u} + \frac{1}{u-2}\, du \\ &= \tfrac{1}{2}\big(\ln|u| + \ln|u-2|\big) + c \\ &= \tfrac{1}{2}\big(\ln(e^x + 2) + \ln(e^x)\big) + c \\ &\equiv \tfrac{1}{2}\big(\ln(e^x + 2) + x\big) + c. \end{aligned}$$

4187. Factorising the LHS,

$$x^2y + xy^2 \equiv xy(x+y).$$

Consider a point (x, y) in the third quadrant, with $x, y < 0$. In this quadrant, xy is positive and $x+y$ is negative. So, $x^2y + xy^2$ is negative and cannot be equal to 1. But the graph shown has points in the third quadrant.

The correct graph for $x^2y + xy^2 = 1$ is



4188. (a) The second derivative is $4e^{2x}$, which is equal to $4y$. So, $y = e^{2x}$ is a solution.

(b) By the product rule,

$$\begin{aligned} y &= \mathrm{f}(x)e^{2x} \\ \Longrightarrow \frac{dy}{dx} &= \big(\mathrm{f}'(x) + 2\,\mathrm{f}(x)\big)e^{2x} \\ \Longrightarrow \frac{d^2y}{dx^2} &= \big(\mathrm{f}''(x) + 4\,\mathrm{f}'(x) + 4\,\mathrm{f}(x)\big)e^{2x}. \end{aligned}$$

Substituting this into the DE,

$$\begin{aligned} &\big(\mathrm{f}''(x) + 4\,\mathrm{f}'(x) + 4\,\mathrm{f}(x)\big)e^{2x} = 4\,\mathrm{f}(x)e^{2x} \\ \Longrightarrow\ &\big(\mathrm{f}''(x) + 4\,\mathrm{f}'(x)\big)e^{2x} = 0. \end{aligned}$$

The exponential factor cannot be zero, so

$$\mathrm{f}''(x) + 4\,\mathrm{f}'(x) = 0.$$

Integrating this with respect to x,

$$\mathrm{f}'(x) + 4\,\mathrm{f}(x) = k, \text{ as required.}$$

(c) Writing $z = \mathrm{f}(x)$,

$$\begin{aligned} \frac{dz}{dx} + 4z &= k \\ \Longrightarrow \frac{dz}{dx} &= k - 4z. \end{aligned}$$

Separating the variables and integrating,

$$\begin{aligned} \int \frac{1}{k-4z}\,dz &= \int 1\,dx \\ \Longrightarrow -\tfrac{1}{4}\ln|k-4z| &= x + c. \end{aligned}$$

Rearranging and renaming constants,

$$\begin{aligned} \ln|k-4z| &= -4x + d \\ \therefore\ \ k - 4z &= Ce^{-4x} \\ \Longrightarrow z &= -\tfrac{1}{4}k - \tfrac{1}{4}Ce^{-4x}. \end{aligned}$$

Renaming the constants again, this is

$$\mathrm{f}(x) = A + Be^{-4x}.$$

(d) Substituting the above back into the original proposed solution,

$$\begin{aligned} y &= \mathrm{f}(x)e^{2x} \\ &= \left(A + Be^{-4x}\right)e^{2x} \\ &\equiv Ae^{2x} + Be^{-2x}, \text{ as required.} \end{aligned}$$

4189. The implication goes forwards. If $\mathrm{f}'(x)$ has a factor of $(x-1)^2$, then

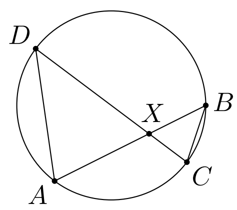
$$\begin{aligned} &\mathrm{f}'(x) = (x-1)^2\,\mathrm{g}(x) \\ \Longrightarrow\ &\mathrm{f}''(x) = 2(x-1) + (x-1)^2\,\mathrm{g}'(x). \end{aligned}$$

The converse doesn't hold. As a counterexample,

$$\begin{aligned} &\mathrm{f}'(x) = (x-1)^2 + 1 \\ \Longrightarrow\ &\mathrm{f}''(x) = 2(x-1). \end{aligned}$$

The latter has a factor of $(x-1)$, but the former doesn't have a factor of $(x-1)^2$.

4190. Consider triangles AXD and BXC:



By the same segment theorem, $\angle ADX = \angle XBC$ and $\angle DAX = \angle XCB$. Also $\angle AXD$ and $\angle BXC$ are opposite and therefore equal. Hence, triangles AXD and BXC are similar. So,

$$\frac{|AX|}{|DX|} = \frac{|CX|}{|BX|}.$$

Rearranging this gives

$$|AX||BX| = |CX||DX|, \text{ as required.}$$

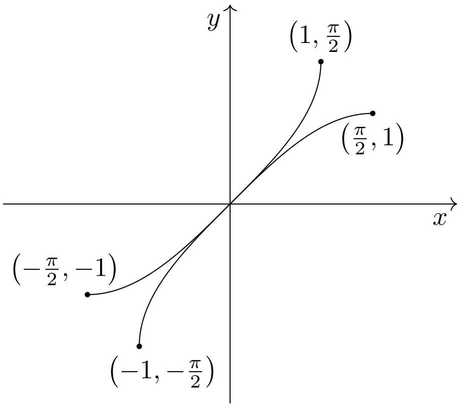
4191. Separating the variables,

$$\begin{aligned} \int x^{-2}\,dx &= \int t^{-1}\,dt \\ \Longrightarrow -x^{-1} &= \ln t + c. \end{aligned}$$

Substituting $t = 1$ and $x = 5$ gives $c = -\frac{1}{5}$. Putting this back in for the particular solution,

$$\begin{aligned} x^{-1} &= \tfrac{1}{5} - \ln t \\ \Longrightarrow x &= \frac{1}{\frac{1}{5} - \ln t} \\ &\equiv \frac{5}{1 - 5\ln t}. \end{aligned}$$

4192. (a) The graphs are



(b) We are looking for the gradient of $y = \arcsin x$ at the point $(\sin x, x)$. Its reflection in $y = x$ is $(x, \sin x)$. At this point, quoting the standard derivative, the gradient of $y = \sin x$ is $\cos x$. Reflecting back to $y = \arcsin x$, gradients are reciprocating by the switching of x and y. This gives

$$f'(\sin x) = \frac{1}{\cos x}, \text{ as required.}$$

(c) Since $\cos x$ is positive over the domain of the arcsin function, we can take the positive square root in $\sin^2 x + \cos^2 x = 1$, replacing $\cos x$ by $\sqrt{1 - \sin^2 x}$. This gives

$$f'(\sin x) = \frac{1}{\sqrt{1 - \sin^2 x}}.$$

Renaming $\sin x$ as x, the above statement is

$$f'(x) = \frac{1}{\sqrt{1 - x^2}}, \text{ as required.}$$

4193. The restricted possibility space is as follows, with classification by (unordered) set of values.

Values	Total	Successful
$(1, 5, 6)$	$3!$	0
$(2, 4, 6)$	$3!$	2
$(2, 5, 5)$	3	0
$(3, 3, 6)$	3	0
$(3, 4, 5)$	$3!$	2
$(4, 4, 4)$	1	1

There are 25 outcomes. Of these, 5 are successful. This gives $\mathbb{P}(X + Z = 2Y \mid X + Y + Z = 12) = \frac{1}{5}$, as required.

4194. The roots of $x \sin x = 0$ are at integer multiples of π. The nth region is formed over the interval $[(n-1)\pi, pi]$. Its area (with a mod sign to deal with the negative signed areas) is

$$A_n = \left| \int_{(n-1)\pi}^{n\pi} x \sin x \, dx \right|.$$

We integrate by parts. Let $u = x$ and $\frac{dv}{dx} = \sin x$. So, $\frac{du}{dx} = 1$ and $v = -\cos x$. This gives

$$\begin{aligned} A_n &= \left| \Big[-x \cos x \Big]_{(n-1)\pi}^{n\pi} + \int_{(n-1)\pi}^{n\pi} \cos x \, dx \right| \\ &= \left| \Big[-x \cos x + \sin x \Big]_{(n-1)\pi}^{n\pi} \right| \\ &= \big| (\pm n\pi) - (\mp(n-1)\pi) \big| \\ &= |\pm(2n-1)\pi| \\ &= (2n-1)\pi. \end{aligned}$$

This is an arithmetic progression, with first term $A_1 = \pi$ and common difference 2π.

4195. Let the equation of the cubic be $y = f(x)$ and the equation of the line be $y = g(x)$. Call the point of inflection $x = \alpha$.

Consider the equation for intersections, which is $f(x) - g(x) = 0$. This is a cubic equation, with a root at $x = \alpha$.

Consider the multiplicity of this root. A tangent at a point of inflection crosses the curve. Hence, $x = \alpha$ must be a triple root of $f(x) - g(x) = 0$. Since the equation is cubic, this leaves no other roots. Hence, the tangent does not intersect the curve again. □

4196. (a) i. The squared magnitudes are

$$\begin{aligned} |\mathbf{a}|^2 = |\mathbf{b}|^2 &= \sec^2 \phi + \tan^2 \phi \\ &\equiv \frac{1 + \sin^2 \phi}{\cos^2 \phi}. \end{aligned}$$

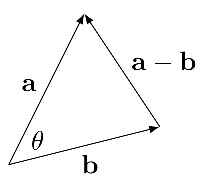
ii. The difference is

$$\begin{aligned} \mathbf{a} - \mathbf{b} &= (\sec \phi - \tan \phi)\mathbf{i} + (\tan \phi - \sec \phi)\mathbf{j} \\ &\equiv (\sec \phi - \tan \phi)(\mathbf{i} - \mathbf{j}). \end{aligned}$$

The squared magnitude of $(\mathbf{i} - \mathbf{j})$ is 2. So,

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= 2(\sec \phi - \tan \phi)^2 \\ &\equiv 2 \left(\frac{1 - \sin \phi}{\cos \phi} \right)^2 \\ &\equiv \frac{2(1 - \sin \phi)^2}{\cos^2 \phi}. \end{aligned}$$

(b) The expressions in part (a) are the lengths of three sides of the triangle formed by vectors **a** and **b**:



Using the cosine rule, and $|\mathbf{a}| = |\mathbf{b}|$,

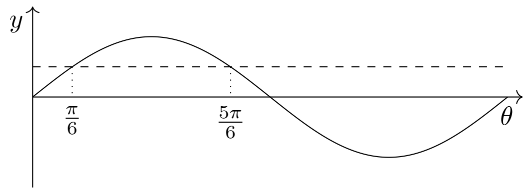
$$\begin{aligned}\cos\theta &= \frac{|\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2}{2|\mathbf{a}||\mathbf{b}|} \\ &= \frac{2|\mathbf{a}|^2 - |\mathbf{a}-\mathbf{b}|^2}{2|\mathbf{a}|^2} \\ &= \frac{2\frac{1+\sin^2\phi}{\cos^2\phi} - 2\frac{(1-\sin\phi)^2}{\cos^2\phi}}{2\frac{1+\sin^2\phi}{\cos^2\phi}}.\end{aligned}$$

Multiplying top and bottom by $\frac{1}{2}\sec^2\theta$, this is

$$\begin{aligned}&\frac{1+\sin^2\phi - (1 - 2\sin\phi + \sin^2\phi)}{1+\sin^2\phi} \\ \equiv\ &\frac{2\sin\phi}{1+\sin^2\phi}, \text{ as required.}\end{aligned}$$

4197. In harmonic form, $\mathrm{f}(\theta) = R\sin(\theta+\alpha)$. The value of R is the Pythagorean sum of 6 and 8, which is 10. We don't need to find α: the possibility space is the interval $[0, 2\pi)$, and, irrespective of the value of α, this interval is exactly one period. So, we can use $\mathrm{g}(\theta) = 10\sin\theta$ wlog. Consider $\theta \in [0, 2\pi)$ as the possibility space.

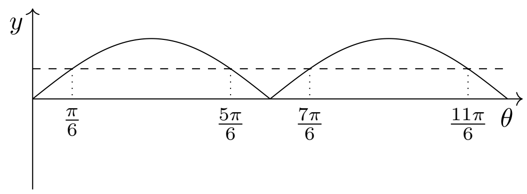
(a) Graphing $y = 10\sin\theta$ and $y = 5$:



The successful interval is $(\pi/6, 5\pi/6)$, which has length $2\pi/3$. This gives

$$\mathbb{P}\big(\mathrm{f}(\theta) > 5\big) = \frac{\frac{2\pi}{3}}{2\pi} = \frac{1}{3}.$$

(b) Graphing $y = |10\sin\theta|$ and $y = 5$:



There are two successful intervals, each as long as that in part (a). So,

$$\mathbb{P}\big(|\mathrm{f}(\theta)| > 5\big) = \frac{\frac{4\pi}{3}}{2\pi} = \frac{2}{3}.$$

4198. The quadratic formula gives

$$\begin{aligned}x_1, x_2 &= \frac{-p \pm \sqrt{p^2 - 4q}}{2}, \\ x_3, x_4 &= \frac{p \pm \sqrt{p^2 - 4q}}{2}.\end{aligned}$$

Adding the two roots of each equation, the square roots cancel, leaving

$$\begin{aligned}x_1 + x_2 &= -p, \\ x_3 + x_4 &= p.\end{aligned}$$

Hence, $x_1 + x_2 + x_3 + x_4 = -p + p = 0$, as required.

—— Alternative Method ——

Consider the graph $y = x^2 + px + q$. Reflecting this in the y axis gives

$$\begin{aligned}&y = (-x)^2 + p(-x) + q \\ \Longrightarrow\ &y = x^2 - px + q.\end{aligned}$$

This shows that the parabolae $x^2 + px + q = 0$ and $x^2 - px + q = 0$ are reflections of one another in $x = 0$. By symmetry, therefore, the sum of the four roots is zero, as required.

4199. (a) The coordinates of the midpoint of the door are $\left(\frac{1}{2}\cos\theta, \frac{1}{2}\sin\theta\right)$. So, by Pythagoras,

$$\begin{aligned}l^2 &= \left(\tfrac{1}{2}\cos\theta\right)^2 + \left(\tfrac{1}{2} - \tfrac{1}{2}\sin\theta\right)^2 \\ &\equiv \tfrac{1}{4}\cos^2\theta + \tfrac{1}{4} - \tfrac{1}{2}\sin\theta + \tfrac{1}{4}\sin^2\theta \\ &\equiv \tfrac{1}{2}(1 - \sin\theta).\end{aligned}$$

So, $\sin\theta = 1 - 2l^2$, as required.

(b) Differentiating implicitly,

$$\cos\theta\frac{d\theta}{dt} = -4l\frac{dl}{dt}.$$

The winch retracts the cable at constant speed u, so $\frac{dl}{dt} = -u$. This gives

$$\begin{aligned}&\cos\theta\frac{d\theta}{dt} = 4lu \\ \Longrightarrow\ &\frac{d\theta}{dt} = \frac{4lu}{\cos\theta} \\ &\qquad = \frac{4lu}{\sqrt{1-\sin^2\theta}}.\end{aligned}$$

Substituting $\sin^2\theta = \left(1 - 2l^2\right)^2$,

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{4lu}{\sqrt{1 - (1 - 4l^2 + 4l^4)}} \\ &\equiv \frac{4lu}{\sqrt{4l^2 - 4l^4}} \\ &\equiv \frac{2u}{\sqrt{1 - l^2}}, \text{ as required.}\end{aligned}$$

(c) The speed of opening is maximised when the denominator is minimised, which occurs at the maximum possible length l. So, the greatest angular speed of opening occurs initially.

4200. The constant of integration has appeared too late. It should appear at the moment the last integral has been enacted:

$$\begin{aligned} & \int e^y \, dy = \int 2x + 1 \, dy \\ \Longrightarrow\ & e^y = x^2 + x + c \\ \Longrightarrow\ & y = \ln(x^2 + x + c). \end{aligned}$$

——— End of 42nd Hundred ———